1. Let us begin with a simple compound interest problem. If \( P_0 \) dollars is invested at \( r\% \) annual interest for \( t \) years, you obtain the exponential equation \( P = P_0 (1+r)^t = P_0 a^t \) where \( a \) is the growth factor and \( r \) is the annual growth rate.

We want to distinguish between an annual growth rate and a continuous growth rate.

2. Suppose you compounded your interest \( k \) times a year, instead of annually. In this case you divide the interest rate by \( k \) (i.e. use \( r/k \) for the interest rate) but multiply the number of times you compound by \( k \) (i.e. \( k \cdot t \) for the exponent). This yields the equation \( P = P_0 \left(1 + \frac{r}{k}\right)^{kt} \)

3. If \( k = 4 \) (i.e. \( P = P_0 \left(1 + \frac{r}{4}\right)^{4t} \)) you’re compounding quarterly; if \( k = 12 \) (i.e. \( P = P_0 \left(1 + \frac{r}{12}\right)^{12t} \)) you’re compounding monthly; if \( k = 365 \) (i.e. \( P = P_0 \left(1 + \frac{r}{365}\right)^{365t} \)) you’re compounding daily. But what about compounding hourly? Or compounding minutely? Or even compounding every second? This raises the question, what is the limit as \( k \) approaches infinity? (compounding continuously). In other words, what is \( \lim_{k \to \infty} P_0 \left(1 + \frac{r}{k}\right)^{kt} \)? This limit is what we call compounding continuously.

4. To evaluate this limit let \( x = \frac{k}{r} \) (or equivalently \( \frac{r}{k} = \frac{1}{x} \)). Obviously \( k \) goes to infinity if and only if \( x \) goes to infinity. Hence the limit \( \lim_{k \to \infty} P_0 \left(1 + \frac{r}{k}\right)^{kt} \) can be re-written as

\[
\lim_{x \to \infty} P_0 \left(1 + \frac{1}{x}\right)^{x r t} = P_0 \left( \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x \right)^t \]

Notice the limit \( \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x \) is 0.

5. By definition \( e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x \) so \( P_0 \left( \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x \right)^t = P_0 e^{rt} \). So in equations of this form, \( r \) is the continuous growth (interest) rate.

6. There is faster growth with continuous compounding versus annual compounding – but not by much. Likewise continuous exponential growth is faster than annual exponential growth – but not by much.

7. **Example**: $100.00 invested at 10% interest compounded annually earns $100.00(1 + 0.1)^1 = 110.00 after one year. The same investment compounded continuously after one year equals $100.00e^{0.1\cdot1} = 110.52$. Thus continuous compounding is a little faster than annual compounding – but not by much.