Witt Sem 100: Fractals and Chaos – Finding Hidden Order

Properties of Chaos: Sensitivity – Mixing – Periodicity

1. The Sawtooth (Doubling Function) $S(x)=\begin{cases} 2x \text{ if } 0 \le x < 0.5 \\ 2x-1 \text{ if } 0.5 \le x \le 1 \end{cases}$

Called the Sawtooth function because of the shape of its graph, the effect of S(x) is to multiply by 2 then *subtract* the integer part if there is one. It is easy to demonstrate that every rational seed is *eventually* periodic. Its behavior is similar to the Tent function.

Example: 5/7 -> S(5/7) = 3/7 -> S(3/7) = 6/7 -> S(6/7) = 5/7 which has period 3

Example: 0.1 -> S(0.1) = 0.2 -> S(0.2) = 0.4 -> S(0.4) = 0.8 -> S(0.8) = 0.6 -> S(0.6) = 0.2 which is *eventually periodic* with period 4

Example: 2/5 -> S(2/5) = 4/5 -> S(4/5) = 3/5 -> S(3/5) = 1/5 -> S(1/5) = 2/5 which has period 4.

Example: 3/16 -> S(3/16) = 6/16 -> S(6/16) = 12/16 -> S(12/16) = 8/16 -> S(8/16) = 0 -> S(0) = 0 which is a *eventually periodic* with period 1 (*eventual fixed point*). Observe that every fraction with a denominator equal to 2^n is an *eventual fixed point*.

On the TI-83 the Sawtooth function can be easily calculated using the `fPart()` (fractional part) function:

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Y₁ = fPart(2X)*((0 ≤ X) and (X ≤ 1.0))
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2. Binary Representation for Values on [0, 1]

Every number x_0 on the interval [0,1] can be represented as an (infinite) sum of powers of ½ in much the same way that x_0 has a decimal expansion in powers of 1/10.

Example: $\frac{1}{4} = 0\times\frac{1}{2^1} + 1\times\frac{1}{2^2} + 0\times\frac{1}{2^3} + \ldots$

Example: $\frac{1}{3} = 0\times\frac{1}{2^1} + 1\times\frac{1}{2^2} + 0\times\frac{1}{2^3} + 1\times\frac{1}{2^4} + 0\times\frac{1}{2^5} + 1\times\frac{1}{2^6} + \ldots$

We *abbreviate* this representation by just displaying the digits without the powers of 1/2 so 1/4 = 0.010000… and 1/3 = 0.010101010101…. In general $x_0 = 0.a_1a_2a_3\ldots a_n\ldots$ where each a_n is either 0 or 1.

If you have a repeating sequence, the geometric series formula $\sum_{k=1}^{\infty} ar^k = a\frac{r}{1-r}$ for $|r|<1$ can be used to convert binary representations to their decimal equivalents.

Example $0.\overline{0110} = \frac{6}{16} + \frac{6}{16^2} + \frac{6}{16^3} + \ldots = 6\frac{1/16}{1-1/16} = \frac{6}{15} = \frac{2}{5}$

3. The Sawtooth function and Binary Representations

The effect of the Sawtooth function S(x) on a binary representation is a **left shift** since the *effect of the Sawtooth function is to multiply by 2 then drop the integer part.* That is

$$s(x_0) = s(0.a_1a_2a_3a_4...) = 0.a_2a_3a_4...$$

Therefore **any repeating binary representation** is periodic under S(x). That is, for $x_0 = 0.\overline{a_1a_2a_3...a_n}$, $S^{[n]}(x_0) = S^{[n]}(0.\overline{a_1a_2a_3...a_n}) = 0.\overline{a_1a_2a_3...a_n}$ because we left shifted n times!

Example: We showed that $\frac{2}{5} = 0.\overline{0110}$. If we apply S( ) to $0.\overline{0110}$ we have a period 4 sequence

$$0.\overline{0110} \to S(0.\overline{0110}) = 0.\overline{1100} \to S(0.\overline{1100}) = 0.\overline{1001} \to S(0.\overline{1001}) = 0.\overline{0011} \to S(0.\overline{0011}) = 0.\overline{0110}$$

Note the *rotate effect* of the Sawtooth function applied to repeating binary sequences; you should verify that this works since it is not difficult to show that the intermediate values $0.\overline{1100}$ etc. represent 4/5, 3/5 and 1/5.

Example $0.\overline{1100} = \frac{12}{16} + \frac{12}{16^2} + \frac{12}{16^3} + ... = 12\frac{1/16}{1-1/16} = \frac{12}{15} = \frac{4}{5}$

4. Periodic points for S(x) are dense on [0,1].

a. Every point on [0,1] has a binary representation $x_0 = 0.a_1\ a_2\ a_3\ a_4\ \dots\ a_n\ \dots$although it is not unique (e.g. 0.01111.. is same as 0.10000) so we disallow representations that end in all 1's.

b. Representations which differ in the n+1st positions are less then $\frac{1}{2^n}$ units apart. That is if

$x_0 = 0.a_1a_2a_3...a_na_{n+1}...$ and $x_1 = 0.a_1a_2a_3...a_na^*_{n+1}...$ then $|x_0 - x_1| < \frac{1}{2^n}$ since the representation of $x_0 - x_1$ is n zeros followed by a 1 (assuming $x_0 - x_1 > 0$).

Example: If $x_0 = 0.11011$ and $x_1 = 0.11010$ then $x_0 - x_1 = 0.11011 - 0.11010 = 0.00001 = \frac{1}{2^5} < \frac{1}{2^4}$

c. Given any point $x_0 = 0.a_1a_2a_3...a_na_{n+1}...$ the periodic point $x_p = 0.\overline{a_1a_2...a_n}$ is within $\frac{1}{2^n}$ units of it. Thus the periodic points for the Sawtooth function are dense on [0,1]

5. The Sawtooth function and sensitivity to initial conditions

Let $x = 0.a_1a_2a_3...a_na_{n+1}a_{n+2}...$ and let $w = 0.a_1a_2a_3...a_na'_{n+1}a'_{n+2}...$. where a'_{n+1} is the *complement* or *dual* of a_{n+1} ; i.e. if $a_n = 0$ *then* $a_n' = 1$ and if $a_n = 1$ *then* $a_n' = 0$. As we saw above the difference between x and w is $|x - w| < \frac{1}{2^n}$ which we can make as small as we want by choosing n large. However, after iterating x and w n times with the Sawtooth function, we have

$$|S^{[n]}(0.a_1a_2a_3...a_na_{n+1}a_{n+2}..) - S^{[n]}(0.a_1a_2a_3...a_na'_{n+1}a'_{n+2}..)| = |0.a_{n+1}a_{n+2}.. - 0.a'_{n+1}a'_{n+2}..| \geq |0.100..| = \frac{1}{2}$$

since a_{n+1} and a'_{n+1} are *complements.*

Therefore given two points, x and w, arbitrarily close ($\frac{1}{2^n}$ is very small), after n iterations, the distance between $S^{[n]}(x)$ and $S^{[n]}(w)$ is greater than $\frac{1}{2}$. This is sensitivity to initial condition.

Example: If $x = 0.11011$ and $w = 011010$ then initially $| x - w] = 0.00001 = \frac{1}{2^5} = \frac{1}{32}$. However after four iterations with the Sawtooth function $| S^{[4]}(x) - S^{[4]}(w) |=| 0.1 - 0.0 |= \frac{1}{2}$

6. The Sawtooth function and mixing

Given two intervals [a,b] and [c,d], there is seed x in [a,b] and an integer n such that $S^{[n]}(x)$ is in [c,d]. We'll show this using an example

Example. Consider the two intervals $[\frac{3}{63}, \frac{4}{64}]$ and $[\frac{36}{64}, \frac{37}{64}]$. It's not difficult to show the midpoints are respectively $\frac{7}{128} = 0.0000111$ and $\frac{73}{128} = 0.1001001$. Let $x = 0.000011101001001$ (do you see the pattern?). Since $\left|\frac{7}{128} - x\right| < \frac{1}{2^8} = \frac{1}{256}$ because $\frac{7}{128}$ and x differ in the 9th place, it follows that x is an element of the interval $[\frac{3}{63}, \frac{4}{64}]$. However $S^{[8]}(x) = 0.1001001 = \frac{73}{128}$ which is contained in $[\frac{36}{64}, \frac{37}{64}]$.

7. The Tent Map: $T(x) = \begin{cases} 2x \text{ if } 0 \le x \le 0.5 \\ 2(1-x) \text{ if } 0.5 < x \le 1 \end{cases}$

The behavior of the Tent map is similar to the Sawtooth function. Note that $2(1-x) = 2 - 2x$

8. Result: T(T(x)) = T(S(x)).

Proof by direct calculation of four cases: 0 ≤x≤ 1/4, 1/4 < x ≤ 1/2, 1/2 < x ≤ 3/4, and 3/4< x ≤1.

Case 1: $0 \le x \le \frac{1}{4}$: trivially $T(T(x)) = T(S(x))$

Case 2: $\frac{1}{4} \le x \le \frac{1}{2}$:

$$T(T(x)) = T(2x) = 2 - 4x$$
$$T(S(x)) = T(2x) = 2 - 4x$$

Case 3: $\frac{1}{2} \le x \le \frac{3}{4}$:

$$T(T(x)) = T(2 - 2x) = 2 - 2(2 - 2x) = 4x - 2$$
$$T(S(x)) = T(2x - 1) = 2(2x - 1) = 4x - 2$$

Case 4: $\frac{3}{4} \le x \le 1$

$$T(T(x)) = T(2 - 2x) = 2 - 2(2 - 2x) = 4x - 2$$
$$T(S(x)) = T(2x - 1) = 2(2x - 1) = 4x - 2$$

Corollary: In general it follows that $T^{[n+1]}(x) = T(S^{[n]}(x))$

9. Tent Map Periodic Points

Let w_0 be periodic with period n for S(x); i.e. $S^{[n]}(w_0) = w_0$. Let $x_0 = T(w_0)$. Then x_0 is periodic for T with period n.

Proof: $T^{[n]}(x_0) = T^{[n]}(T(w_0)) = T(S^{[n]}(w_0)) = T(w_0) = x_0$

Example: 5/7 is periodic with period 3 for S. T(5/7) = 4/7. Hence 4/7 is period with period 3 for T.

Check: 4/7 -> T(4/7) = 2(1-4/7) = 6/7 -> T(6/7) = 2(1 – 6/7) = 2/7 -> T(2/7) = 4/7 for period 3.

10. Tent Map and Binary Representation

There is a simple way to understand the action of the Tent map T(x) on binary representations

$T(0.a_1a_2a_3...) = \begin{cases} 0.a_2a_3a_4...\ if\ a_1 = 0 \\ 0.a_2'a_3'a_4'...\ if\ a_1 = 1 \end{cases}$ where recall we defined $a_n' = \begin{cases} 1\ if\ a_n = 0 \\ 0\ if\ a_n = 1 \end{cases}$ - called the *complement* or *dual*. Note that the *dual of a dual* is the original; i.e. $a_n'' = a_n$.

The lower case rule (complement and left shift) is easily seen since for $\frac{1}{2} \le x \le 1$, the action of the Tent function is to subtract x from 1 (thus the complement) and multiply by 2 (thus the left shift).

Example: $T\left(\frac{5}{7}\right) = T(0.\overline{101}) = 2\left(1 - \frac{5}{7}\right) = \frac{4}{7} = 0.\overline{100}$. Observe that you start with $0.\overline{101}$, complement to obtain $0.\overline{010}$ and left shift to obtain $0.\overline{100}$.

11. Tent Map Periodic Points are Dense on [0,1]

a. Any point in [0,1] can be closely approximated by $x_0 = 0.a_1a_2a_3...a_n$

b. We've shown any repeating binary representation $0.\overline{0a_1a_2a_3...a_n}$ is periodic with period n+1 under S(x)

c. Therefore claim $0.\overline{a_1a_2a_3...a_n0} = T(0.\overline{0a_1a_2a_3...a_n})$ is periodic with period n+1 under T(x)

Example: To show the density of period points, w/log start with some interval whose width is a power of $\frac{1}{2}$, for example [5/16, 6/16]. Find the midpoint which in this example is 11/32. In binary this is 0.01011. The repeating binary $0.\overline{001011}$ (note additional leading 0) is periodic under S(x) with period 6. Applying the Tent function we obtain $T(0.\overline{001011}) = 0.\overline{010110}$ which is periodic under T with period 6.

$0.\overline{010110} = \sum_{k=1}^{\infty} \frac{22}{64^k} = \frac{22/64}{1 - 1/64} = \frac{22}{63}$ which is also on the interval [5/16, 6/16]

Check: 22/63 -> T(22/63)=44/63 -> T(44/63) = 38/63 -> T(38/63) = 50/63 -> T(50/63) = 26/63 -> T(26/63) = 52/63 -> T(52/63) = 22/63 for period 6.

Note that the midpoint is $\frac{11}{32}$ and the close periodic point is $\frac{22}{63}$. The method is summarized (and simplified) below

Method for finding dense periodic points

1. Starting with the interval $\left[\frac{i}{2^{n-1}}, \frac{i+1}{2^{n-1}}\right]$ the midpoint will be $\frac{k}{2^n}$ where k is odd.
2. The point $\frac{2k}{2^{n+1}-1}$ will be a periodic point in the interval.

12. Tent Map Mixing and Sensitivity

Demonstrations of mixing and sensitivity properties for the Tent map are similar to the demonstrations for the Sawtooth function.

Sensitivity: For sensitivity to initial conditions, find two seeds x and w whose binary representations agree in the first n positions but differ in the n+1st. The two seeds are within $\frac{1}{2^n}$ units of each other (i.e. close together for some large n) but after n iterations of the Tent function, the distance will increase to $\frac{1}{2}$.

Mixing: The technical details for showing mixing are more difficult but essentially follow what was done in the case of the Sawtooth function. Given two intervals [a, b] and [c, d] find the midpoints x and w for each interval and use their binary representations to construct the binary representation of a point in [a,b] which after n iterations is in [c,d].

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