The Verhulst Population Dynamics Equation

Unbounded population growth equations are simple to describe mathematically. If x_n is the population after the n[th] year and R is the *relative increase* per year, i.e.

$$R = \frac{x_{n+1} - x_n}{x_n}$$

then if the R is a constant value, the population after n years can be computed to be

$$x_n = (1+R)^n x_0$$

This equation can be refined by assuming that a) the environment is only able to sustain a certain size population say X (normalized to 1) and b) the growth rate R is a linear function of the "distance" between the current population x_n and 1, that is

$$R = c(1 - x_n)$$

where c is called the *growth factor*. Combining the first and third equations yields the Verhulst Population Equation

$$x_{n+1} = (1+c)x_n - cx_n^2$$

Note: The Verhulst equation is *similar* to the logistic equation $x_{n+1} = rx(1-x)$. It has roots at 0 and $\frac{(1+c)}{c}$ and its vertex is the point $\left(\frac{1+c}{2c}, \frac{(1+c)^2}{4c}\right)$. Since $\frac{(1+c)^2}{4c} \leq \frac{1+c}{2c}$ on the interval (0, 3), the Verhulst equation maps the interval $\left[0, \frac{1+c}{2c}\right]$ onto itself.

The behavior of the Verhulst Population Equation as a function of c:

1. For $0 < c < 2$: attracting fixed point at $x = 1$
2. For $2 \leq c < 2.57$: equation undergoes a series of period doublings
3. $2.57 < c \leq 3.0$: chaotic behavior!

Verhulst Programs see Q:\Mathematics\Classes\WittSem\Verhulst\
Graphfcn.exe - graph of function & higher order iterates; web diagrams
Bifurcat.exe - bifurcation diagram
Verhulst.xls - Verhulst time series: shows sensitivity to initial conditions

Exploring the Verhulst Equation $x_{n+1} = (1+c)x_n - cx_n^2$ using the TI-83 with parameter C

To clear graph use [Draw] ClrDraw
To display both table and graph use [Mode] G-T

1. Generating Time Series Value in Seq Mode

[Mode] Seq
[Format] Time
[Y=] $\eta Min = 1$
$\mu(\eta) = (1+C)\mu(\eta-1) - C\mu(\eta-1)^2$ note: μ is [2nd][7]; use [X,T,θ,η] key for η
$\mu(\eta Min) = 0.5$
[Table]

2. Displaying Graph of Time Series in Seq Mode

[Window] – set appropriate values for ηMin and ηMax, Xmin and Xmax, Ymin and Ymax
[Graph]

3. Displaying a Web Diagram in Seq Mode

[Format] Web
[Window] Xmin = 0, Xmax > $(1+c)/2c$
Ymin = 0, Ymax > $(1+c)^2/4c$
[Graph]
[Trace] [->]

4. Function Mode Calculations

[Mode] Func
[Y =] $Y_1 = (1+C)X - CX^2$
$Y_2 = X$ <- for y = x line

To generate Time Series use

Initialize C, initialize X e.g 2.5 [STO->] C and 0.5 [STO->] X)

Y1(X) [STO->] X note Y1 is obtained from [Vars] Y-Vars 1:Function (enter) 1:Y1 (enter)