The Verhulst Population Dynamics Equation

Unbounded population growth equations are simple to describe mathematically. If \( x_n \) is the population after the \( n \)th year and \( R \) is the relative increase per year, i.e.

\[
R = \frac{x_{n+1} - x_n}{x_n}
\]

then if the \( R \) is a constant value, the population after \( n \) years can be computed to be

\[
x_n = (1 + R)^n x_0
\]

This equation can be refined by assuming that a) the environment is only able to sustain a certain size population say \( X \) (normalized to 1) and b) the growth rate \( R \) is a linear function of the "distance" between the current population \( x_n \) and 1, that is

\[
R = c(1 - x_n)
\]

where \( c \) is called the growth factor. Combining the first and third equations yields the Verhulst Population Equation

\[
x_{n+1} = (1 + c)x_n - cx_n^2
\]

Note: The Verhulst equation is similar to the logistic equation \( x_{n+1} = rx(1 - x) \). It has roots at 0 and \( \frac{1+c}{c} \) and its vertex is the point \( \left( \frac{1+c}{2c}, \frac{(1+c)^2}{4c} \right) \). Since \( \frac{(1+c)^2}{4c} \leq \frac{1+c}{2c} \) on the interval \((0, 3)\), the Verhulst equation maps the interval \( [0, \frac{1+c}{2c}] \) onto itself.

The behavior of the Verhulst Population Equation as a function of \( c \):

1. For \( 0 < c < 2 \): attracting fixed point at \( x = 1 \)
2. For \( 2 \leq c < 2.57 \): equation undergoes a series of period doublings
3. \( 2.57 < c \leq 3.0 \): chaotic behavior!

Verhulst Programs see Q:\Mathematics\Classes\WittSem\Verhulst\ Graphfcn.exe - graph of function & higher order iterates; web diagrams Bifurcat.exe - bifurcation diagram Verhulst.xls - Verhulst time series: shows sensitivity to initial conditions
Exploring the Verhulst Equation $x_{n+1} = (1 + c)x_n - cx_n^2$ using the TI-83 with parameter C

To clear graph use [Draw] ClrDraw
To display both table and graph use [Mode] G-T

1. Generating Time Series Value in Seq Mode

[Mode] Seq
[Format] Time
[Y=] \( \eta Min = 1 \)
\[ \mu(\eta) = (1 + C)\mu(\eta - 1) - C\mu(\eta - 1)^2 \text{ note: } \mu \text{ is } [2nd][7]; \text{ use } [X,T,\theta,\eta] \text{ key for } \eta \]
\[ \mu(\eta Min) = 0.5 \]
[Table]

2. Displaying Graph of Time Series in Seq Mode

[Window] – set appropriate values for \( \eta Min \) and \( \eta Max \), Xmin and Xmax, Ymin and Ymax
[Graph]

3. Displaying a Web Diagram in Seq Mode

[Format] Web
[Window] \( Xmin = 0, Xmax > \frac{(1 + c)}{2c} \)
\[ Ymin = 0, Ymax > \frac{(1 + c)^2}{4c} \]
[Graph]
[Trace] [->]

4. Function Mode Calculations

[Mode] Func
[Y=] \( Y_1 = (1 + C)X - CX^2 \)
\[ Y_2 = X \text{ <- for } y = x \text{ line} \]

To generate Time Series use

Initialize C, initialize X e.g 2.5 [STO->] C and 0.5 [STO->] X

\( Y1(X) \) [STO->] X note Y1 is obtained from [Vars] Y-Vars 1:Function (enter) 1:Y1 (enter)